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Decomposition of δ-Continuity and δ^*-Continuity

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Abstract: In the present paper, the notion of δ-open sets, g-closed sets, δg-closed sets; and the relation between them has been studied. It is also noted that the collection of δ-open sets form the topology. A new concept of δ^*-continuity has been established which is a generalization of the classical form of continuity. By introducing the idea of δ-fine ope n set and δg-fine open set, δ-fine continuity and δg-fine continuity have been defined. In support of these new concepts, several illustrative examples have been given.

Keywords: δ-open sets, g-closed sets, δg-closed sets.
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I. Introduction

If any set is contained in each member of the collection of some open sets with the property that its closure is also contained in each member of the same collection, then the set is named (cf. [1]) as generalized closed set and some basic properties of these sets have been also investigated in [1]. The characterization of a locally closed set which is the intersection of an open set with the closed set has been explored by Bourbaki [2]. Ganster and Reilly [3] had extended the work of Bourbaki and initiated the concept of LC-continuous functions to establish the decomposition of classical continuous functions (see also [4], [5], [6]).

Recently, Ekici [7] has introduced the notions of several generalized sets viz. δ-semi-generalized closed sets, locally δ-semi-generalized closed sets, $l\delta sgc^*$, $l\delta sgc^{**}$-sets. It has been studied in [7] that the class of δ-semi-generalized closed sets contain the classes of δg-closed sets and δ-semi closed sets, also the properties of these classes are investigated. The main objective of this paper (cf. [7]) is to introduce and study an extended form of locally closed sets (which are called generalized locally closed sets or glc-sets). On the basis of glc-sets, several generalized forms of continuities have been defined in [7].

In the present work, it has been noticed that the collection of all δ-open sets constitute a topology and using this property, the idea of δ-fine open set, δg-fine open set have been introduced. In accordance with these generalized fine open sets, the notion of some new continuous functions is also defined and studied.

II. Preliminaries

Throughout this paper, (X,τ) and (Y,σ) denote topological spaces on which no separation axioms are assumed unless stated otherwise. F_X and F_Y denote collection of closed sets corresponding to the topologies on X and Y respectively. For a subset A ⊆ X, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$ respectively. A function $f: X \to Y$ denotes a single valued function of a topological space (X,τ) into topological space (Y,σ). τ_f denotes the collection of fine open sets generated by the topology τ on X and σ_f denotes the collection of fine-open sets generated by the topology σ on Y.

The following definitions and the concepts are required for establishing the assertions of the present paper:

Definition 2.1. A subset S of a space (X, τ) is called

- **Pre-open** [8] if S ⊆ int(cl(S))
- **Semi-open** [9] if S ⊆ cl(int(S).
- **α-open** [8] if S ⊆ int(cl(int(S))).
- **β-open** [10] if S ⊆ cl(int(cl(S)).
- **Regular-open** [7] if S = int(cl(S)).

The complement of pre-open set (semi-open set, α-open set, β-open set and regular-open set) is called pre-closed set (semi-closed set, α-closed set, β-closed set and regular-closed set) respectively. The collection of α-open sets (semi-open sets, pre-open sets, β-open sets and regular-open sets) denoted by αO(X) (SO(X), PO(X), βO(X) and RO(X)) respectively.

Remark 2.1. The following inclusions are the direct consequences of the definitions given above:

- $\boldsymbol{\tau \subseteq \alpha O(X) \subseteq SO(X) \subseteq \beta O(X)}$
- $\boldsymbol{\tau \subseteq \alpha O(X) \subseteq PO(X) \subseteq \beta O(X)}$

Definition 2.2. [7] A subset A of a space (X, τ) is said to be **δ-open** if $\forall\ x \in A\ \ \exists$ a regular-open set B such that $x \in B \subseteq A$. The collection of **δ-open** sets of X is denoted by **δo(X).** The complement of δ-open set is **δ-closed** set. The collection of δ-closed sets of X is denoted by **δc(X)** .

Example 2.1. Consider X = {a, b, c, d} with the topology τ = {X, ϕ, {b}, {c}, {b, c}, {a, b}, {a, b, c}}. F_X = {ϕ, X, {a, c, d}, {a, b, d}, {a, d},{c, d},{d}}. In view of Definition 2.1, we have RO(X) = {X, ϕ, {a, b}, {c}}. Consider A = {a, b, c} a subset of X.

Claim: A = {a, b, c} is δ-open.

- A point a ∈ {a, b, c} ∃ a regular-open set {a, b} such that a ∈{a, b} ⊆ {a, b, c}.(cf. Definition 2.2)
- Again a point b ∈ {a, b, c} ∃ a regular-open set {a, b} such that b ∈ {a, b} ⊆ {a, b, c}.
- A point c ∈ {a, b, c} ∃ a regular-open set {c} such that c ∈ {c} ⊆ {a, b, c}.

Hence A = {a, b, c} is **δ-open set**. The complement of {a, b, c} is {d} which is a **δ-closed set**.

Definition 2.3. [7] A subset A of a space (X, τ) is said to be

- **g-closed** if cl(A) ⊆ U whenever A⊆ U, $U \in \tau$. The collection of g-closed sets is denoted by **gc(X)**
- **δg-closed** if δcl(A) ⊆ U whenever A ⊆ U, $U \in \tau$. The collection of δg-closed sets is denoted by **δgc(X)**.

The complement of g-closed set (δg-closed set) is called **g-open** (**δg-open**) set respectively. The collection of g-open (δg-open) sets of X is denoted by **go(X)** (**δgo(X)**) respectively.

Example 2.2. Consider a topological space X = {a, b, c, d} with the topology τ = {X, ϕ, {a, b, c}}, F_X = {ϕ, X, {d}}. Consider A = {a, d} a subset of X. There is only one open set say U = X containing A. Then it is easy to check that cl{a, d}= X which follows by the definition that cl{a, d} = X ⊆ X = U. Hence, A = {a, d} is **g-closed**.

Example 2.3. Consider a topological space X = {a, b, c, d} with the topology τ = {X, ϕ, {c}, {d}, {c, d}, {a, c}, {a, c, d}}, F_X = {ϕ, X, {a, b, d}, {a, b, c}, {a, b}, {b, d}, {b}}. In view of Definition 2.2, we have δo(X) = {X, ϕ, {a, c}, {d}, {a, c, d}}, δ_{F_X} = {X, ϕ, {b, d}, {a, b, c}, {b}} where δ_{F_X} is a collection of δ-closed sets of X. Consider A = {b, c} be a subset of X. Since there is only one open set say U = X containing A. Then according to the Definition 2.3, δcl(A) = δcl{b, c} = {a, b, c} ⊆ X = U. Thus A ={b, c} is **δg-closed set**.

Remark 2.2. The following implications are the direct consequences of the definitions:

- **δc(X)** ⇒ **F_X** ⇒ **gc(X)**
- **δo(X)** ⇒ **τ** ⇒ **go(X)**

Example 2.4. Consider a topological space X = {a, b, c, d} with the topology τ = {X, ϕ, {a, c, d}, {b, c, d}, {c, d}}, F_X = {ϕ, X, {b}, {a}, {a, b}}. In view of Definition 2.3, the collection of g-closed sets and δg-closed sets are gc(X) = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}, δgc(X) ={X, ϕ, {a, b, d},{a, b, c},{a, b}} respectively. It is easy to see that the class of g-closed sets contains the class of δg-closed sets and the class of closed sets as well. Moreover, the class of closed sets is independent of the class of δg-closed sets.

Remark 2.3. The following implication (cf. Figure 2.1) may be verified from the definitions:

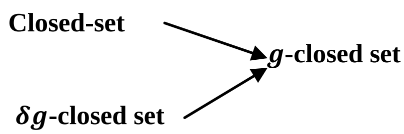

Figure 2.1

Definition 2.4. [11] Let X be a nonempty set and g be a collection of subsets of X. Then **g** is called a **generalized topology** (briefly GT) on X if $\phi \in$ g and $G_i \in$ g for i ∈ I ≠ ϕ implies G = $\cup_{i \in I} G_i \in$ g. We say **g** is **strong** if X ∈ g; and we call the pair (X, g) a generalized topological space on X. The elements of g are called **g-open** sets and their complements are called **g-closed** sets.

Example 2.5. Let X = {a, b, c} then, we may define following generalized topologies on X.

$\mathbf{g_1}$ = {ϕ, {a}}

$\mathbf{g_2}$ = {ϕ, {a}, {b}, {a, b}}

$\mathbf{g_3}$ = {ϕ, X, {a, c}, {a, b}}

In this Example $\mathbf{g_1}$, $\mathbf{g_2}$ are the generalized topologies but not strong generalized topologies (cf. Definition 2.4) whereas $\mathbf{g_3}$ is not a topology but it is a strong generalized topology.

Definition 2.5. [12] Let (X, τ) be a topological space we define, $\tau\,(A_\alpha)$ = τ_α= $\{G_\alpha\,(\neq X) : G_\alpha \cap A_\alpha \neq \phi,$ for $A_\alpha \in \tau$ and $A_\alpha \neq X, \phi$ for some $\alpha \in J$, where J is the index set}. Now, define $\tau_f = \{\phi, X\} \cup \{\tau_\alpha\}$. The above collection $\boldsymbol{\tau_f}$ of subsets of X is called the **fine collection** of subsets of X and (X, τ, τ_f) is said to be the **fine space** of X generated by the topology τ on X.

Example 2.6. Consider a topological space X = {a, b, c} with the topology τ = {X, ϕ, {a}} ≡ {X, ϕ, A_α} where $A_\alpha = \{a\}$. In view of Definition 2.5, we have $\tau_\alpha = \tau\,(A_\alpha) = \tau\,\{a\}$ = {{a},{a, b},{a, c}}. Then the **fine collection** is $\tau_f = \{\phi, X\} \cup \{\tau_\alpha\}$ = {ϕ, X, {a}, {a, b}, {a, c}}.

We refer some important properties of fine topological spaces.

Lemma 2.1. [12] Let $(X, \tau, \tau_f\,)$ be a fine space then arbitrary union of fine open sets in X is fine-open set in X.

Lemma 2.2. [12] The intersection of two fine-open sets need not be a fine-open set.

Example 2.7. Let X= {a, b, c} be a topological space with the topology τ ={X, ϕ, {a}, {b}, {a, b}}, τ_f = {X, ϕ, {a}, {b}, {a, b}, {b, c}, {a, c}}. It may be seen that, the collection τ_f is not a topology. Hence, the collection of fine open sets in a space X does not form a topology on X, but it is a strong generalized topology on X.

Remark 2.4. In view of Definition 2.4 of generalized topological space and above Lemmas 2.1 and 2.2, it is apparent that (X, τ, τ_f) is a special case of generalized topological space. It may be noted specifically that the topological space plays a key role while defining the fine space as it is based on the topology of X but there is no topology in the back of generalized topological space.

Definition 2.6. [12] A subset U of a fine space X is said to be a **fine-open set** of X, if U belongs to the collection τ_f and the complement of every fine open set of X is called the **fine-closed set** of X and the collection of fine-closed sets is denote by $\boldsymbol{F_f}$.

Remark 2.5. [12] The family of all α-open sets respectively (β-open sets, pre-open sets, semi-open sets) is denoted by $\big(\alpha O(X), \beta O(X), PO(X),\ SO(X)\big)$. It has been concluded in [12] that

- $\boldsymbol{\tau \subseteq \alpha O(X) \subseteq SO(X)) \subseteq \beta O(X) \subseteq \tau_f}$
- $\boldsymbol{\tau \subseteq \alpha O(X) \subseteq PO(X)) \subseteq \beta O(X) \subseteq \tau_f}$

Definition 2.7. [12] Let A be a subset of a fine space X the **fine-interior** of A, is defined as the union of all fine-open sets contained in the set A ie. the largest fine-open set contained in the set A and it is denoted by $\boldsymbol{f_{int}}$.

Example 2.8. Let X = {a, b, c} be a topological space with the topology τ = {X, ϕ, {a}, {b}, {a, b}} then the fine collection is τ_f = {X, ϕ, {a}, {b}, {a, b}, {b, c}, {a, c}}. We can see that, int{a, c}={a} and f_{int}{a, c} = {a, c}.

Definition 2.8. [12] Let A be a subset of a fine space X the **fine-closure** of A, is defined as the intersection of all fine-closed sets containing the set A and it is denoted by $\boldsymbol{f_{cl}}$.

Example 2.9. Let X = {a, b, c} be a topological space with the topology τ = {X, ϕ {a}}, F_X = {X, φ, {b, c}}, then the fine collection is τ_f = {X, ϕ, {a}, {a, b}, {a, c}}, F_f = {φ, X, {b, c}, {c}, {b}}.We can see that, $cl\{b\}$ ={b, c} and $f_{cl}\{b\} = \{b\}$.

III. Main Result

In this section, it has been established that the class of δ-open sets form the topology.

Theorem 3.1. Let (X,τ) be the topological space and δo(X) be the collection of all δ-open sets in X. Then the collection δo(X) forms the topology on X.

For the proof of this theorem, we require the following Lemma.

Lemma 3.1. Let (X,τ) be the topological space and RO(X) be the collection of all regular open sets in X. Then the intersection of two regular-open sets is a regular-open set.

Proof of the Lemma 3.1. Let B_1 and B_2 be two regular -open sets. Then by the definition, we have

$$B_1 = \text{int}(\text{cl}(B_1));\ \ B_2 = \text{int}(\text{cl}(B_2)) \qquad (3.1)$$

It is enough if we show the following

$$B_1 \cap B_2 = \text{int}(\text{cl}(B_1 \cap B_2)) \qquad (3.2)$$

Since,

$$B_1 \cap B_2 \subseteq B_1 \Rightarrow \text{cl}(B_1 \cap B_2) \subseteq \text{cl}(B_1)$$

$$\Rightarrow \text{int}(\text{cl}(B_1 \cap B_2)) \subseteq \text{int}(\text{cl}(B_1)) \qquad (3.3)$$

Again,

$$B_1 \cap B_2 \subseteq B_2 \Rightarrow \text{cl}(B_1 \cap B_2) \subseteq \text{cl}(B_2)$$

$$\Rightarrow \text{int}(\text{cl}(B_1 \cap B_2)) \subseteq \text{int}(\text{cl}(B_2)) \qquad (3.4)$$

Using (3.1), (3.3), (3.4) we get,

$$\text{int}(\text{cl}(B_1 \cap B_2)) \subseteq B_1 \cap B_2 \qquad (3.5)$$

On the other hand, as

$B_1 \cap B_2 \subseteq \mathrm{cl}(B_1 \cap B_2)$ (3.6)

Since, B_1 and B_2 are open, then their intersection is also open. Hence,

$B_1 \cap B_2 = \mathrm{int}(B_1 \cap B_2)$ (3.7)

Taking int on both sides of (3.6), we get

$\mathrm{int}(B_1 \cap B_2) \subseteq \mathrm{int}(\mathrm{cl}(B_1 \cap B_2))$ (3.8)

Using (3.7) and (3.8) we have,

$B_1 \cap B_2 \subseteq \mathrm{int}(\mathrm{cl}(B_1 \cap B_2))$ (3.9)

Thus, in view of (3.5) and (3.9), we conclude the following:

$B_1 \cap B_2 = \mathrm{int}(\mathrm{cl}(B_1 \cap B_2)$

This completes the proof of Lemma 3.1.

Proof of the Theorem 3.1. Let A be a δ-open set. Then in view of Definition 2.2, we have

δo(X) = {A : ∀ x ∈ A ∃ a regular-open B such that x ∈ B ⊆ A}.

Claim: δo(X) is a topology on X.

- X ∈ δo(X) since, ∀ x ∈ X ∃ a regular-open set X such that x ∈ X ⊆ X.
- ϕ ∈ δo(X) since, there is no element in ϕ, Hence, this condition holds vacuously.

Consider the collection $\{A_\alpha\}_{\alpha\in J}$ of δ-open sets belonging to the class of δo(X).

Claim: A = $\cup (A_\alpha)_{\alpha\in J}$ ∈ δo(X).

It is enough if we show that ∀ x ∈ A ∃ a regular-open G such that x ∈ G ⊆ A.

Consider x ∈ A ⇒ x ∈ $\cup_{\alpha\in J} A_\alpha$ ∃ some β ∈ J such that x ∈ A_β where A_β ∈ δo(X), for some β ∈ J. It implies that ∃ a regular-open set G such that x ∈ G ⊆ A_β ⊆ $\cup A_\alpha$ = A (cf. Definition 2.2). Hence, the arbitrary union of members of δo(X) is in δo(X).

Claim: A = $A_1 \cap A_2$ ∈ δo(X).

Since, A_1 is δ-open ⇒ ∃ a regular-open B_1 such that,

$x \in B_1 \subseteq A_1$ (3.10)

Similarly, A_2 is also δ-open set, it follows that ∃ a regular-open set B_2 such that,

$x \in B_2 \subseteq A_2$ (3.11)

Using (3.11) and (3.12) we have,

$x \in B_1 \cap B_2 \subseteq A_1 \cap A_2$ (3.12)

It is a direct consequence of the Lemma 3.1 that the intersection of two regular-open sets is a regular-open set, hence $B_1 \cap B_2$ is regular open set. Finally we conclude that the class of δ-open sets form the topology. This completes the proof of Theorem 3.1.

IV. δ^*-Continuity

In this section, we define a new class of generalized continuous functions, which is the extension of the concept of δ-continuous function initiated by Ekici [7].

Definition 4.1. [7] A function $f:(X,\tau)\to(Y,\sigma)$ is said to be **δ-continuous** if $f^{-1}(G)$ is δ-open set in X for each $G\in\sigma$.

Example 4.1. Consider X = {a, b, c, d} and Y = {1, 2, 3, 4} be the two topological spaces with their corresponding topologies are τ = {X, ϕ, {a}, {b},{a, b},{a, c},{a, b, c}}, σ = {Y, ϕ, {2, 3}, {2}, {1, 2, 3}}. Then in view of Definition 2.2, we have δo(X) = {X, ϕ, {b}, {a, c}, {a, b, c}}. We now define a function $f: X \to Y$ as:

$$f(x)=\begin{cases}2\ for\ x=a\ or\ c\\ 3\ for\ x=b\\ 4\ for\ x=d\end{cases} \quad (4.1)$$

In view of (4.1) for $G\in\sigma$, we get

$$f^{-1}(G)=\begin{cases}X\ for\ G=Y\\ \phi\ for\ G=\phi\\ \{a,c\}\ for\ G=\{2\}\\ \{a,b,c\}\ for\ G=\{2,3\}\\ \{a,b,c\}\ for\ G=\{1,2,3\}\end{cases} \quad (4.2)$$

Hence, we notice that $f^{-1}(G)\in\delta$o(X) and in view of Definition 4.1, we conclude that f is **δ-continuous**.

Definition 4.2. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be **δ^*-continuous** if $f^{-1}(V)$ is open in X for every δ-open set V of Y.

Example 4.2. Consider X = {a, b, c, d} and Y = {1, 2, 3, 4} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a}, {b, c}, {a, b, c}, {b, c, d}}, σ = {Y, ϕ, {1}, {2}, {1, 2}, {2, 3}, {1, 2, 3}}. In view of Definition 2.2, we have δo(Y) = {Y, ϕ,{1},{2,3},{1,2,3}}.We now define a function $f: X \to Y$ as:

$$f(x)=\begin{cases}1\ for\ x=a\\ 2\ for\ x=b\\ 3 for\ x=c,d\end{cases} \quad (4.3)$$

In view of (4.3) for $G\in\delta$o(Y), we get

$$f^{-1}(G)=\begin{cases}X\ for\ G=Y\\ \phi\ for\ G=\phi\\ \{a\}\ for\ G=\{1\}\\ \{b,c,d\}\ for\ G=\{2,3\}\\ X\ for\ G=\{1,2,3\}\end{cases} \quad (4.4)$$

Hence, we notice that $f^{-1}(G)\in\tau$ and conclude that f is **δ^*-continuous**, (cf. Definition 4.2), moreover it may be checked that f is neither **continuous** nor **δ-continuous function**.

Remark 4.1. The following implications are the direct consequences of Definition 4.1 and Definition 4.2 (see also [13] page no. 102).

δ-Continuity $\Rightarrow$ Continuity $\Rightarrow$ δ^*-Continuity

V. δ-Fine Set

Using the concept of fine-open sets (cf. Definitions 2.5, 2.6), we are now in a position to define the idea of δ-fine and δg-fine sets.

Definition 5.1. Let (X,τ) be a topological space. We define $(\delta_\alpha) = \delta(A_\alpha) = \{G_\alpha\ (\neq X) : G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \delta$o(X) and $A_\alpha \neq X, \phi$ for some $\alpha \in J$, where J is the index set and δo(X) is a collection of δ-open sets of $X\}$. Now, define $\delta_f(X) = \{\phi, X\} \cup \{\delta_\alpha\}$. The above collection $\delta_f(X)$ of subsets of X is called the **δ-fine collection** of subsets of X and $(X, \delta o(X), \delta_f(X))$ is said to be the **δ-fine space** of X generated by the topology τ on X. The elements of $\delta_f(X)$ are called **δ-fine open sets** of X.

Example 5.1. Consider a topological space X = {a, b, c, d} with the topology τ = {X, ϕ, {a}, {b}, {a, b}, {a, c}, {a, b, c}}. In view of Definition 2.2, we have δo(X)={X,ϕ,{b},{a, c}, {a, b, c}} $\equiv \{X, \phi, A_\alpha, A_\beta, A_\gamma\}$
where A_α= {b}, A_β={a, c}, A_γ={a, b, c}
(see Definition 5.1). Then, we have
$(\delta_\alpha) = \delta(A_\alpha) = \delta$\{b\} = {{b},{a, b},{b, c},{b, d},{a, b, c},{a, b, d}, {b, c, d}}
$(\delta_\beta) = \delta(A_\beta) = \delta$\{a, c\}={{a},{a, b} ,{c}, {a, c}, { a, d}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, b, d}}
$(\delta_\gamma) = \delta(A_\gamma) = \delta$\{a, b, c\} = {{a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, b, d}} (see Definition 5.1).
Then the **δ-fine collection is $\delta_f(X)$** = $\{\phi, X\} \cup \{\delta_\alpha\} \cup \{\delta_\beta\} \cup \{\delta_\gamma\}$ = {ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}.

Definition 5.2. Let (X,τ) be a topological space. We define $(\delta g_\alpha) = \delta g(A_\alpha) = \{G_\alpha\ (\neq X) : G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \delta g$o(X) and $A_\alpha \neq X, \phi$ for some $\alpha \in J$, where J is the index set and δgo(X) is a collection of δg-open sets of $X\}$. Now, define $\delta g_f(X)$ =$\{\phi, X\} \cup \{\delta g_\alpha\}$. The above collection $\delta g_f(X)$ of subsets of X is called the **δg-fine collection** of subsets of X and $(X, \delta go(X), \delta g_f(X))$ is said to be the **δg-fine space** of X generated by the topology τ on X. The elements of $\delta g_f(X)$ are called **δg-fine open sets** of X.

Example 5.2. Consider a topological space X = {a, b, c, d} with the topology τ = {X, ϕ, {a, c, d}, {b, c, d}, {c, d}} In view of Definition 2.3, we have δgo(X) ={X, ϕ, {c}, {d}, {c, d,}} $\equiv \{X, \phi, A_\alpha, A_\beta, A_\gamma\}$ where A_α={c}, A_β={d}, A_γ={c, d} (see Definition 5.2). Then, we have
$(\delta g_\alpha) = \delta g(A_\alpha) = \delta g$\{c\} = {{c}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}}
$(\delta g_\beta) = \delta g(A_\beta) = \delta g$\{d\}={{d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}
$(\delta g_\gamma) = \delta g(A_\gamma) = \delta g$\{c, d\} = {{c}, {d}, {c, d}, {a, d}, {a, c}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}} (see Definition 5.2).
Then the **δg-fine collection is $\delta g_f(X)$** = $\{\phi, X\} \cup \{\delta g_\alpha\} \cup \{\delta g_\beta\} \cup \{\delta g_\gamma\}$ = {ϕ, X, {c}, {d}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}.

VI. δ-Fine Continuity

In this section, the extension of fine-continuous functions has been investigated.

Definition6.1.[12] A function $f: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is said to be **fine-continuous** if $f^{-1}(V)$ is open set in X for every fine-open set V of Y.

Example 6.1. Consider X = {a, b, c} and Y = {1, 2, 3} be the two topological spaces with their corresponding topologies τ ={X, ϕ, {a}, {b, c}}, σ ={Y, ϕ,{1, 3}}. Then in view of Definition 2.5, we have σ_f = {Y, ϕ, {1}, {3}, {1, 2}, {1, 3}, {2, 3}}. Define a function $f: X \to Y$ as:

$$f(x) = \begin{cases} 1 & for\ x = a \\ 2 & for\ x = b, c \end{cases} \qquad (6.1)$$

In view of (6.1) for G ∈ σ_f, we get

$$f^{-1}(G) = \begin{cases} X\ for\ G = Y \\ \phi\ for\ G = \phi \\ \{a\}\ for\ G = \{1\} \\ \phi\ for\ G = \{3\} \\ \{a\} for\ G = \{1,3\} \\ X\ for\ G = \{1,2\} \\ \{b, c\}\ for\ G = \{2,3\} \end{cases} \qquad (6.2)$$

Hence, we notice that $f^{-1}(G) \in \tau$. Therefore f is **fine-continuous.**

Definition 6.2. [14] A function $f: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is called **fine super-continuous** if $f^{-1}(G)$ is fine-open in X for each G ∈ σ.

Example 6.2. Consider the topological spaces X = {a, b, c, d}, Y = {1, 2, 3, 4} with their corresponding topologies τ ={X, ϕ, {a}, {a, b}, {a, b, c}}, σ = {Y, ϕ, {1}, {2}, {1, 2}}. In view of Definition 2.5, we have τ_f = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. We now define the function $f: X \rightarrow Y$ as:

$$f(x) = \begin{cases} 1 \ for \ x = a \\ 2 \ for \ x = b \\ 3 \ for \ x = c \\ 4 \ for \ x = d \end{cases} \qquad (6.3)$$

In view of (6.3) for $G \in \sigma$, we get

$$f^{-1}(G) = \begin{cases} X \ for \ G = Y \\ \phi \ for \ G = \phi \\ \{a\} \ for \ G = \{1\} \\ \{b\} \ for \ G = \{2\} \\ \{a, b\} \ for \ G = \{1,2\} \end{cases} \qquad (6.4)$$

Hence, we notice that $f^{-1}(G) \in \tau_f$ and conclude that f is **fine super-continuous** when we appeal the Definition 6.2, but it can be verified that f is neither **fine-continuous** nor **continuous**.

Remark 6.1. In view of the above definitions and examples of different continuous functions, the following implications are direct:

Fine-continuity ⇒ Continuity ⇒ Fine Super-Continuity

We now define the concepts of δ-fine continuous function, δg-fine continuous function, δ-fine super-continuous function and δg-fine super-continuous function.

Definition 6.3. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be ***δ*-fine-continuous** if $f^{-1}(G)$ is open set in X for each $G \in \delta_f(Y)$.

Example 6.3. Consider X = {a, b, c, d} and Y = {1, 2, 3, 4} be the two topological spaces with their corresponding topologies τ ={X, ϕ, {a}, {b}, {a, b}, {a, c}, {a, b ,c}, {a, c, d}}, σ = {Y, ϕ, {2}, {1, 4}, {2, 4}, {4}, {1, 2, 4}}. In view of Definition 2.2, the collection of δ-open sets of Y is $\delta o(Y)$ ={X, ϕ, {2}, {1, 4}, {1, 2, 4}}. Then the collection of δ-fine-open sets of Y is $\delta_f(Y)$ = {Y, ϕ, {1}, {2}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}. We now define a function $f: X \rightarrow Y$ as:

$$f(x) = \begin{cases} 1 \ for \ x = a \ or \ c \ or \ d \\ 2 \ for \ x = b \end{cases} \qquad (6.5)$$

In view of (6.5) for $G \in \delta_f(Y)$, we get $f^{-1}(G) \in \tau$ and referring Definition 6.3, we conclude that f is ***δ*-fine-continuous**.

Definition 6.4. A function $f: (X, \tau, \tau_f) \rightarrow (Y, \sigma)$ is said to be ***δ*-fine super-continuous** if $f^{-1}(G)$ is δ-fine-open set in X for each $G \in \sigma$.

Example 6.4. Consider X = {a, b, c, d} and Y = {1, 2, 3, 4} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a}, {c}, {a, c}, {b, c}, {a, b, c}}, σ = {Y, ϕ, {1}, {1, 3}, {1, 4}, {1, 3, 4}}. Then in view of Definition 2.2 and 5.1, the collections of δ-open sets and δ-fine-open sets of X are δo(X) = {X, ϕ, {a}, {b, c}, {a, b, c}}, δ_f(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c},{a, b, d}, {a, c, d} ,{b, c, d}} respectively. We now define a function $f: X \rightarrow Y$ as:

$$f(x) = \begin{cases} 1 \ for \ x = a \ or \ c \\ 2 \ for \ x = b \\ 4 \ for \ x = d \end{cases} \qquad (6.6)$$

In view of (6.6) for $G \in \sigma$, we get

$$f^{-1}(G) = \begin{cases} X \ for \ G = Y \\ \phi \ for \ G = \phi \\ \{a, c\} \ for \ G = \{1\} \\ \{a, c\} \ for \ G = \{1,3\} \\ \{a, c, d\} \ forG = \{1,4\} \\ \{a, c, d\} \ for \ G = \{1,3,4\} \end{cases} \qquad (6.7)$$

Hence, we notice that $f^{-1}(G) \in \delta_f(X)$ and conclude that f is ***δ*-fine super-continuous**, (cf. Definition 6.4), but it is neither **continuous** nor ***δ*-continuous**.

Definition 6.5. [7] A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be ***δg*-continuous** if $f^{-1}(G)$ is δg-open set in X for each $G \in \sigma$.

Example 6.5. Consider X = {a, b, c} and Y = {1, 2, 3} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a}}, σ = {Y, ϕ, {1}, {2}, {1, 2}}. Then in view of Definition 2.3, the collections of δg-open sets of X is δgo(X) = {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}}. We now define a function $f: X \rightarrow Y$ as:

$$f(x) = \begin{cases} 1 \ for \ x = a \\ 2 \ for \ x = b \\ 3 \ for \ x = c \end{cases} \qquad (6.8)$$

In view of (6.8) for $G \in \sigma$, we get

$$f^{-1}(G) = \begin{cases} X \ for \ G = Y \\ \phi \ for \ G = \phi \\ \{a\} \ for \ G = \{1\} \\ \{b\} \ for \ G = \{2\} \\ \{a, b\} for \ G = \{1,2\} \end{cases} \qquad (6.9)$$

Hence, we notice that $f^{-1}(G) \in \delta go(X)$ and conclude that f is ***δg*-continuous** but it is neither **continuous** nor ***δg*-fine-continuous** (cf. Definitions 6.5, 6.6).

Definition 6.6. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be ***δg*-fine-continuous** if $f^{-1}(G)$ is open in X for each $G \in \delta g_f(Y)$.

Example 6.6. Consider X = {a, b, c} and Y = {1, 2, 3} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a}, {b}, {a, b}, {b, c}}, σ = {Y, ϕ, {1, 2}, {2, 3}, {2}}. Then in view of Definition 2.3 and 5.2, the collections of δg-open sets and δg-fine-open sets of Y are $\delta go(Y)$ = {Y,ϕ,{2}}, $\delta g_f(Y)$ = {Y, ϕ, {2}, {1, 2}, {2, 3}} respectively. We now define a function $f: X \to Y$ as:

$$f(x) = \begin{cases} 1 \; for \; x = a \\ 2 \; for \; x = b \\ 3 \; for \; x = c \end{cases} \qquad (6.10)$$

In view of (6.10) for $G \in \delta g_f(Y)$, we get

$$f^{-1}(G) = \begin{cases} X \; for \; G = Y \\ \phi \; for \; G = \phi \\ \{a, b\} \; for \; G = \{1,2\} \\ \{b, c\} \; for \; G = \{2,3\} \\ \{b\} for \; G = \{2\} \end{cases} \qquad (6.11)$$

Hence, we notice that $f^{-1}(G) \in \tau$ and conclude that f is $\boldsymbol{\delta g}$**-fine-continuous** (cf. Definition 6.6), but it is not **fine-continuous** (cf. Definition 6.1).

Definition 6.7. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be $\boldsymbol{\delta g}$**-fine super-continuous** if $f^{-1}(G)$ is δg-fine open set in X for each $G \in \sigma$.

Example 6.7. Consider a topological space X = {a, b, c, d} and Y = {1, 2, 3, 4} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a, c, d}, {b, c, d}, {c, d}} σ = {Y, ϕ, {1, 3}, {1, 4}, {1}, {1, 3, 4}}. In view of Definition 2.3 and 5.2, the collection of δg-open sets and δg-fine-open sets of X are δgo(X) ={X, ϕ, {c}, {d}, {c, d}}, $\delta g_f(X)$ = {X, ϕ, {c}, {d}, {c, d}, {a, d}, {b, d}, {b, c}, {a, c}, {a, b, c}, {a, b, d} ,{a, c, d}, {b, c, d}} respectively. We now define a function $f: X \to Y$ as:

$$f(x) = \begin{cases} 1 \; for \;\; x = a \; or \; c \\ 2 \; for \; x = b \\ 4 \; for \; x = d \end{cases} \qquad (6.12)$$

In view of (6.12) for $G \in \sigma$, we get

$$f^{-1}(G) = \begin{cases} X \; for \; G = Y \\ \phi \; for \; G = \phi \\ \{a, c\} \; for \; G = \{1\} \\ \{a, c\} \; for \; G = \{1,3\} \\ \{a, c, d\} for \; G = \{1,4\} \\ \{a, c, d\} \; for \; G = \{1,3,4\} \end{cases} \qquad (6.13)$$

Hence, we notice that $f^{-1}(G) \in \delta g_f(X)$ and conclude that f is $\boldsymbol{\delta g}$**-fine super-continuous** (cf. Definition 6.6), but it is neither **continuous** nor $\boldsymbol{\delta g}$**-continuous** (cf. Definition 6.5).

Definition 6.8. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be **regular-fine-continuous** if $f^{-1}(G)$ is fine-open set in X for each $G \in RO(Y)$.

Example 6.8. Consider X = {a, b, c} and Y = {1, 2, 3} be the two topological spaces with their corresponding topologies τ = {X, ϕ, {a}}, σ = {Y, ϕ, {1}, {2}, {1, 2}}. Then in view of Definition 2.1 and 2.5, we have RO(Y) = {Y, ϕ, {1}, {2}}, τ_f = {X, ϕ, {a},{a, c}, {a, b}}. We now define a function $f: X \to Y$ as:

$$f(x) = \begin{cases} 3 \; for \; x = b \\ 1 \; for \; x = a, c \end{cases} \qquad (6.14)$$

In view of (6.14) for $G \in RO(Y)$, we get

$$f^{-1}(G) = \begin{cases} X \; for \; G = Y \\ \phi \; for \; G = \phi \\ \{a, c\} \; for \; G = \{1\} \\ \phi \; for \; G = \{2\} \end{cases} \qquad (6.15)$$

Hence, we notice that $f^{-1}(G) \in \tau_f$ and conclude that f is **regular-fine-continuous** (cf. Definition 6.8) but it may be verified that f is not **continuous**.

Remark 6.2. It is a direct consequence of the above definitions and examples that:

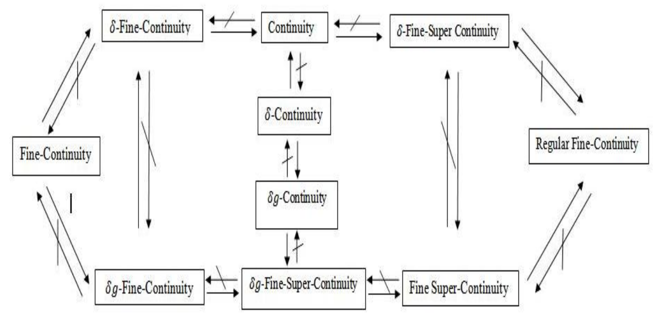

Figure 6.1

VII. Conclusion

The concept of δ^*-continuity is lighter concept of continuity which assigns open set of the domain to the δ-open sets of the range. The situation where it is not possible to define homeomorphism, the δ^*-continuity may be applied to define the corresponding lighter form of δ^*-homeomorphism. A similar concept of homeomorphism can be extended for other continuous functions also. Most of the problems of Quantum Physics are dealt with the idea of homeomorphism and this concept can cover the wider range of such problems.

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