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Ciric Fixed Point Theorems in T- Orbitally Complete Spaces with n-quasi contraction

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Abstract—Poom Kuman, [Poom Kuman , Nguyen van Dung, A generalization of Ciric Fixed Point theorems, Filomat 29:7 (2015), 1549-1556] has established the generalized version of the result by Ciric [L. B. Ciric, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974) 267-273.]. By considering the most general form of quasi-contraction viz. **n-quasi contraction,** the authors have established the existence of unique fixed point in T- orbitally complete spaces in this paper.

Keywords—Fixed Point, n-quasi contraction, T-Orbitally Complete space.

I. INTRODUCTION

Ciric generalized the Banach's contraction principle

[1] by defining the quasi-contraction map in 1974 and proved Ciric Fixed Point theorem. In 2008, Berinde [2] defined ciric-type almost contractions in metric spaces and established existence of fixed point. Lakshmikantham et al in 2009 [3] proved coupled fixed point theorem for nonlinear contraction considering partially ordered metric space. In 2017, Poom Kuman [4] defined generalized quasi-contraction by adding the factors viz. $\{d(T^2x,\ x)$, $d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\}$ in quasi-ciric contraction [1] and proved the generalized result of Ciric [1].

In the present paper, we have defined more general quasi contraction by adding

$$d(T^3x,x), d(T^3x,Tx), d(T^3x,y), d(T^3x,Ty)$$

in generalized quasi contraction due to Poom Kuman [4] and established the existence of Fixed Point in T - Orbitally complete Metric Space. Further, for any positive integer n, we have defined the most extended form of generalized quasi-contraction named as n -quasi contraction dark by considering the condition:

$$\begin{aligned} d(Tx,Ty) \\ \leq q.\max\{ d(x,y), d(x,Tx), d(y,Ty), \\ d(y,Tx), d(x,Ty), d(T^2x,x), \\ d(T^2x,y), d(T^2x,Ty), d(T^2x,\ Tx), \\ d(T^3x,Tx),\ d(T^3x,y), d(T^3x,Ty), \\ d(T^3x,Ty), \dots\dots\dots, d(T^nx,\ x), \\ d(T^nx,Tx),\ d(T^nx,y), d(T^nx,Ty)\}. \end{aligned}$$

It is interesting to note that for n = 1 our definition turns out to be the quasi-contraction due to Ciric. Also n = 2 gives the contraction introduced by Poom Kuman [4].

II. PRELIMINARIES

In order to prove the main result of the paper, we need the following definitions and notions.

Let (X, d) be the metric space and E, F be any two subsets of X then

$$D(E,F) = inf\{d(a,b): a \in E\ , b \in F\}$$
$$\rho(E,F) = sup\{d(a,b): a \in E\ , b \in F\}$$
$$\delta(E) = sup\{d(a,b): a\ , b \in E\}$$

Definition 1[5]. Let $T: X \to X$ be a map on metric space (X, d). For each $x \in X$ and for any positive integer n, denote $O_T(x,\ n) = \{ x, Tx, \dots, T^nx\}$ and $O_T(x, +\infty) = \{ x, Tx, \dots, T^nx, \dots\}$.

The set $O_T(x,+\infty)$ is called the **orbit of T** at x and the metric space X is called **T- Orbitally complete** if every Cauchy sequence in $O_T(x,+\infty)$ is convergent in X.

Example 1: Let (R, d) be metric space with respect to usual metric d, and $T: R \to R$, $T(x) = \frac{x}{4}$. Then orbit of T is $O_T(x,+\infty) = \left\{ x, \frac{x}{4}, \ldots, \frac{x}{4^n}, \ldots \right\}$ and it may be verified easily that R is T-Orbitally complete.

Definition 2[1] : Let $T: X \to X$ be a mapping on metric space (X, d). The mapping T is said to be a **quasi-contraction** if there exists $q \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le q.\max\{ d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\}.$$

Example 2: Let (E, d) be metric space with respect to usual metric d, where E = $[0,\infty)$ and

$$T: [0,\infty) \to [0,\infty), T(x) = \frac{x}{2}.$$

It is clear that T satisfies quasi-contraction condition.

Definition 3 [4]: Let $T: X \to X$ be a mapping on metric space (X, d). The mapping T is said to be a **generalized quasi-contraction** if there exists $q \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le q.\max\{ d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty), d(T^2x,x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\}$$

Example 3: Let (E, d) be metric space with respect to usual metric d, where E = $[0,\infty)$ and $T: [0,\infty) \to [0,\infty)$, $T(x) = \frac{x}{3}$. Then T satisfies generalized quasi contraction condition.

Referring the definition 3, we now introduce the generalized form of quasi-contraction due to Poom Kuman [4].

Definition 4: Let $T: X \to X$ be a mapping on metric space (X, d). The mapping T is said to be a **3-quasi contraction** if there exists $q \in [0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le q.\max\{ d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty), d(T^2x, x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty), d(T^3x,x), d(T^3x,Tx), d(T^3x,y), d(T^3x,Ty)\}.$$

Example 4: Let (R, d) be metric space with respect to usual metric d, and $T: R \to R$, $T(x) = \frac{x}{8}$.

T is a 3-quasi contraction map.

Definition 5: Let $T: X \to X$ be a mapping on metric space (X, d). The mapping T is said to be a **n-quasi contraction** if there exists $q \in [0,1)$ such that for all $x, y \in X$ and $n \in Z^+$,

$$d(Tx,Ty) \le q.\max\{ d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty), d(T^2x,x), d(T^2x,y), d(T^2x,Ty), d(T^2x, Tx), d(T^3x,Tx), d(T^3x,y), d(T^3x,Ty), d(T^3x,Ty), \ldots\ldots\ldots, d(T^nx, x), d(T^nx,Tx), d(T^nx,y), d(T^nx,Ty)\}.$$

Example 5: Let (E, d) be metric space with respect to usual metric d, where E = $[0,\infty)$ and $T: [0,\infty) \to [0,\infty)$, $T(x) = \frac{x}{3}$.

Then T satisfies n-quasi contraction condition.

III. MAIN RESULT

In this section, we state one of the two main results of this paper.

Theorem 1: Let (X, d) be the metric space and $T: X \to X$ be a 3-quasi contraction map (cf. Definition 4). Also X is T- orbitally complete. Then T has a unique fixed point x^* *in* X.

Proof. We first establish the existence of a fixed point under the map T.

For each $x \in X$ and $1 \le i \le n-2$ and $1 \le j \le n$, where $n \in Z^+$

Consider

$$\begin{aligned} d(T^ix,T^jx) &= d(TT^{i-1}x, TT^{j-1}x) \\ &\le q.\max\{ d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, TT^{i-1}x), d(T^{j-1}x, TT^{j-1}x), d(T^{j-1}x, TT^{i-1}x), d(T^{i-1}x, TT^{j-1}x), \\ &\quad d(T^2T^{i-1}x, T^{i-1}x), d(T^2T^{i-1}x, TT^{i-1}x), d(T^2T^{i-1}x, T^{j-1}x), d(T^2T^{i-1}x, TT^{j-1}x), \\ &\quad d(T^3T^{i-1}x, T^{i-1}x), d(T^3T^{i-1}x, TT^{i-1}x), d(T^3T^{i-1}x, T^{j-1}x), d(T^3T^{i-1}x, TT^{j-1}x)\} \\ &\le q.\max\{ d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^ix), d(T^{j-1}x, T^jx), d(T^{j-1}x, T^ix), d(T^{i-1}x, T^jx), \\ &\quad d(T^{i+1}x, T^{i-1}x), d(T^{i+1}x, T^ix), d(T^{i+1}x, T^{j-1}x), d(T^{i+1}x, T^jx), \\ &\quad d(T^{i+2}x, T^{i-1}x), d(T^{i+2}x, T^ix), d(T^{i+2}x, T^{j-1}x), d(T^{i+2}x, T^jx)\} \\ &\le q.\delta[O_T(x,n)] \end{aligned}$$

Where,

$\delta[O_T(x,n)] = \max\{d(T^ix,T^jx) : 0 \le i,j \le n\}$.

Since $0 \le q < 1$, $\exists\ k_n(x) \le n$ such that

$$d\left(x, T^{k_n(x)}x\right) = \delta[O_T(x,n)] \qquad (1)$$

Now, $d\left(x, T^{k_n(x)}x\right) \le d(x,Tx) + d\left(Tx, T^{k_n(x)}x\right)$

$$\leq d(x,Tx) + q.\delta[O_T(x,n)]$$
$$\leq d(x,Tx) + q.d\left(x,T^{k_n(x)}x\right)$$

It implies that

$$(1-q)d\left(x,T^{k_n(x)}x\right) \leq d(x,Tx)$$
$$d\left(x,T^{k_n(x)}x\right) \leq \frac{1}{(1-q)}d(x,Tx)$$

Using (1), we get

$$\delta[O_T(x,n)] = d\left(x,T^{k_n(x)}x\right) \leq \frac{1}{(1-q)}d(x,Tx) \qquad (2)$$

For all $n, m \leq 1$ and $n \leq m$, it follows from 3-quasi contraction condition on T and (2) that

$$d(T^n x, T^m x) = d(TT^{n-1}x, T^{m-n+1}T^{n-1}x)$$
$$\leq q.\delta[O_T(T^{n-1}x,\ m-n+1)]$$
$$\leq q.d(T^{n-1}x, T^{k_{m-n+1}(T^{n-1}x)}T^{n-1}x)$$
$$\leq q.d(TT^{n-2}x, T^{k_{m-n+1}(T^{n-1}x)+1}T^{n-2}x)$$
$$\leq q.d(TT^{n-2}x, T^{k_{m-n+1}(T^{n-1}x)+1}T^{n-2}x)$$
$$\leq q^2.\delta[O_T(T^{n-2}x,\ m-n+2)]$$
$$\leq \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leq q^n.\delta[O_T(x,\ m-n+n)] \quad d(T^n x, T^m x) \leq \frac{q^n}{1-q}d(x,Tx)$$

Since $\lim_{n\to\infty} q^n = 0$

$\{T^n x\}$ is a Cauchy sequence in X. Since X is T- Orbitally complete, $\exists\ x^* \in X$ such that

$$\lim_{n\to\infty} T^n x = x^* \qquad (3)$$

We now show that x^* is a fixed point.

$$d(x^*,Tx^*) \leq d(x^*,\ T^{n+1}x) + d(T^{n+1}x, Tx^*)$$
$$d(x^*,Tx^*) \leq d(x^*,\ T^{n+1}x)$$
$$+ q.\max\{d(T^n x, x^*),\ d(T^n x, T^{n+1}x),$$
$$d(x^*,Tx^*),\ d(T^n x, Tx^*),$$
$$d(x^*,T^{n+1}x),\ d(T^{n+2}x, x^*),$$
$$d(T^{n+2}x, Tx^*),\ d(T^{n+2}x, T^n x),$$
$$d(T^{n+2}x, T^{n+1}x), d(T^{n+3}x, T^n x),$$
$$d(T^{n+3}x, T^{n+1}x),\ d(T^{n+3}x, x^*),$$
$$d(T^{n+3}x, Tx^*)\}$$

As $n \to \infty$ using (3), we get

$$d(x^*,Tx^*) \leq q.\max\{d(x^*,Tx^*)\}$$

Which is possible if

$$d(x^*,Tx^*) = 0$$
$$x^* = Tx^*$$

Hence, it assures the existence of a fixed point x^*.

Claim: x^* is unique.

Let if possible x^*, y^* be two fixed points of T.

$$d(x^*,y^*) = d(Tx^*,Ty^*)$$
$$d(Tx^*,Ty^*) \leq q.\max\{d(x^*,y^*), d(x^*,Tx^*),$$
$$d(y^*,Ty^*), d(y^*,Tx^*), d(Ty^*,x^*),$$
$$d(T^2x^*,x^*), d(T^2x^*,Tx^*),$$
$$d(T^2x^*,y^*), d(T^2x^*,Ty^*),$$
$$d(T^3x^*,x^*), d(T^3x^*,Tx^*),$$

$$d(x^*,y^*) = 0 \implies x^* = y^*$$

Thus, finally, we conclude uniqueness of x^*.

Theorem 2: Let (X, d) be a metric space, $T: X \to X$ be a map satisfying the following conditions

a). X is T- Orbitally complete

b). $d(Tx,Ty) \leq q.\max\{d(x,y), d(x,Tx), d(Ty,x)$
$$d(y,Ty),\ d(y,Tx), d(T^2x,x),$$
$$d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty),$$
$$d(T^3x,\ x), d(T^3x,Tx),\ d(T^3x,y),$$
$$d(T^3x,Ty), \ldots\ldots\ldots, d(T^n x,\ x),$$
$$d(T^n x,Tx),\ d(T^n x,y), d(T^n x,Ty)\}.$$

Then T has a unique fixed point $x^*\ in\ X$.

Proof: In order to prove this tresult, we need the method of mathematical induction.

For n = 1, condition (b) turns out to be quasi contraction defined by Ciric [1] and also the existence and uniqueness of fixed points has been established already. Hence, the result holds for n = 1 which is famous Ciric fixed point theorem.

Consider, T has a unique fixed point in X for n = p, then we have to show that T has a unique fixed point in X for n = p+1. We first establish the existence of a fixed point for n = p+1.

Assuming the condition

$$d(Tx,Ty)$$
$$\leq q.\max\{d(x,y), d(x,Tx), d(y,Ty),$$
$$d(y,Tx), d(x,Ty), d(T^2x,x),$$
$$d(T^2x,y), d(T^2x,Ty), d(T^2x,\ Tx),$$
$$d(T^3x,Tx),\ d(T^3x,y), d(T^3x,Ty),$$
$$d(T^3x,Ty), \ldots\ldots\ldots, d(T^p x,\ x),$$
$$d(T^p x,Tx),\ d(T^p x,y), d(T^p x,Ty)\}.$$

and the existence of unique fixed point for n = p is given.

If T be a map satisfying

$$d(Tx,Ty)$$
$$\leq q.\max\{d(x,y), d(x,Tx), d(y,Ty),$$
$$d(y,Tx), d(x,Ty), d(T^2x,x),$$
$$d(T^2x,y), d(T^2x,Ty), d(T^2x,\ Tx),$$
$$d(T^3x,Tx),\ d(T^3x,y), d(T^3x,Ty),$$
$$d(T^3x,Ty), \ldots\ldots\ldots, d(T^p x,\ x),$$
$$d(T^p x,Tx),\ d(T^p x,y), d(T^p x,Ty),$$
$$d(T^{p+1}x,x),\ d(T^{p+1}x,Tx),$$

$$d(T^{p+1}x, y), d(T^{p+1}x, Ty)\}.$$

Suppose max lies in

$$\{ d(x, y), d(x, Tx), d(y, Ty),$$

$$d(y, Tx), d(x, Ty), d(T^2x, x),$$

$$d(T^2x, y), d(T^2x, Ty), d(T^2x,\ Tx),$$

$$d(T^3x, Tx),\ d(T^3x, y), d(T^3x, Ty),$$

$$d(T^3x, Ty), \ldots\ldots\ldots, d(T^px,\ x),$$

$$d(T^px, Tx),\ d(T^px, y), d(T^px, Ty)\}.$$

then by the given condition T has a unique fixed point.
Suppose max lies between from
$\{d(T^{p+1}x, x), d(T^{p+1}\,x, Tx), d(T^{p+1}x, y), d(T^{p+1}x, Ty)\}$

$$d(x^*, Tx^*) \leq d(x^*, T^rx) + d(T^rx, Tx^*)$$

$$= d(x^*, Tx^r) + q.\max\{d(T^{p+r}x, T^{r-1}x), d(T^{p+r}\,x, T^rx), d(T^{p+r}x, x^*), d(T^{p+r}x, Tx^*)\} \quad (4)$$

Where $r > p$.
Since $\lim_{n\to\infty} T^nx = x^*$.
As $p \to \infty$ in equation (4), we get

$$d(x^*,\ Tx^*) \leq qd(x^*,\ Tx^*).$$

$$d(x^*,\ Tx^*) = 0 \ \Rightarrow\ Tx^* = x^*.$$

Hence, T has a fixed point.
Let if possible x^*, y^* be two fixed points of T.

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$d(Tx^*, Ty^*) \leq\ q.\max\{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), d(y^*, Tx^*), d(x^*, Ty^*), d(T^2x^*, x^*), d(T^2x^*, Tx^*), d(T^2x^*, y^*), d(T^2x^*, Ty^*), \ldots\ldots, d(T^nx^*, x^*), d(T^nx^*, Tx^*), d(T^nx^*, y^*), d(T^nx^*, Ty^*)\}$$

$d(x^*, y^*) = 0 \implies x^* = y^*$ Thus, finally, T has a unique fixed point.

IV. CONCLUSION

The existence and uniqueness of a unique fixed point in T-Orbitally complete space has been established with the most light form of contraction map viz. n-quasi contraction which a significant contribution. This result also extend the domain of Poom Kuman Result.

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