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Continuous generalized Hankel-Clifford wavelet transformation

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Abstract- In this paper, the generalized Hankel-Clifford wavelet transformation is developed. Using the developed theory of generalized Hankel-Clifford convolution, the generalized Hankel-Clifford translation is introduced. Properties of the kernel $D_{\mu,\alpha,\beta}(x,y,z)$ are developed in the study. Using the properties of kernel the generalized Hankel-Clifford wavelet transformation is defined. The existence of the generalized Hankel-Clifford wavelet transformation is given by a theorem. The boundedness and inversion formula for the generalized Hankel-Clifford wavelet transformation is obtained. A basic wavelet which defines continuous generalized Hankel-Clifford wavelet transformation, its admissibility conditions and the wavelet to the function is proved. Examples have been shown to explain the studied continuous generalized Hankel-Clifford wavelet transformation. *MSC:* 44A20, 42C40, 46

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1. Introduction

Malgonde [8] investigated the following generalized Hankel-Clifford transformation

$$F_1(t)=\left(F_{1,\alpha,\beta}f\right)(t)=t^{-\alpha-\beta}\int_0^\infty (xt)^{(\alpha+\beta)/2}\,J_{\alpha-\beta}\left[2\sqrt{xt}\right]f(x)\,dx\ , \tag{1}$$

$J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)\geq -1/2$.

Define $L_p(0,\infty), 1\leq p\leq\infty,$ as the space of real measurable function ϕ on $(0,\infty)$ as studied in [2], for which

$$\|\phi\|_{\beta,p}=\left(\int_0^\infty\left|x^{-\beta}\phi(x)\right|^p\frac{dx}{x}\right)^{1/p}, 1\leq p<\infty. \tag{2}$$

$$\|\phi\|_\infty=\underset{0<x<\infty}{ess\sup}\left|x^{-\beta}\phi(x)\right|<\infty.$$

For each $\phi\in L_1(0,\infty)$, generalized Hankel-Clifford transformation of ϕ is defined by

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$$\hat{\phi}(x) = t^{-\alpha-\beta}\int_0^\infty (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] \phi(t)dt \quad , 0 < t < \infty. \tag{3}$$

From [1], it is observed $\hat{\phi}(x)$ is bounded and continuous on $(0,\infty)$ and $\left\|\hat{\phi}(x)\right\|_\infty \le \|\phi\|_1$.

If $f(x)$ is of bounded variation into a neighborhood of the point $x = x_0 > 0, (\alpha-\beta) \ge -1/2$ and the integral $\int_0^\infty |f(x)| x^{(\alpha+\beta)/2-(1/4)}dx$ exists, then the inversion formula in [8] is given by

$$\lim_{R\to\infty} x_0^{-\alpha-\beta}\int_0^\infty f(x)\int_0^R y^{-\alpha-\beta}(xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xy}\right](x_0y)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x_0y}\right]dydx$$
$$= \frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right] \tag{4}$$

If $f(x)x^\alpha$ and $G_1(y)y^\alpha$ are in $L_1(0,\infty)$,and $(\alpha-\beta) \ge -1/2$ for

$F_1(y) = \left(F_{1,\alpha,\beta} f(x)\right)(y)$ and $g(x) = F^{-1}_{1,\alpha,\beta}\left[G_1(y)\right](x)$, for $(\alpha-\beta) \ge -1/2$, then the following Parseval relation holds for F_1-transformation by [8];

$$\int_0^\infty x^{\alpha+\beta} f(x)g(x)dx = \int_0^\infty y^{\alpha+\beta}F_1(y)G_1(y)dy. \tag{5}$$

If $f(x)x^\alpha$ and $G_2(y)y^\beta$ are in $L_1(0,\infty)$, and $(\alpha-\beta) \ge -1/2$ for

$F_1(y) = F_{1,\alpha,\beta}\left[f(x)\right](y)$ and $G_2(y) = F_{1,\alpha,\beta}\left[g(x)\right](y)$, for $(\alpha-\beta) \ge -1/2$, then the following Mixed Parseval's equation holds for F_1-transformation by [8];

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_1(y)G_2(y)dy. \tag{6}$$

To define the generalized Hankel-Clifford Convolution, there is a need to introduce generalized Hankel-Clifford translation similar to [6]. Define

$$D_{\alpha,\beta}(x,y,z)$$
$$= \int_0^\infty t^{-\alpha}\left\{t^{-\alpha-\beta}(xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right]t^{-\alpha-\beta}(yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right]t^{-\alpha-\beta}(zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right]\right\} dt \tag{7}$$

Properties of the kernel $D_{\alpha,\beta}(x,y,z)$:

Following [3] properties are established:

i) For $0 < x, y < \infty$ and $0 \le t < \infty$, we have

$$\int_0^\infty t^\alpha \left\{ t^{-\alpha-\beta} (zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right]\right\} D_{\alpha,\beta}(x,y,z)\,dz$$
$$= (xy)^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right] \quad (8)$$

Proof:

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty t^{-\alpha} \left\{ t^{-\alpha-\beta} (zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right]\right\} \left\{ \begin{matrix} x^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] \\ y^{-\alpha-\beta} (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right] \end{matrix} \right\} dz$$

$$= \int_0^\infty t^{-\alpha} t^{-\alpha-\beta} \left[\begin{matrix} (xy)^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] \\ (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right] \end{matrix} \right] (zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right] dz$$

$$= t^{-\alpha} F_{1,\alpha,\beta}\left\{ (xy)^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right]\right\}$$

$$F^{-1}_{1,\alpha,\beta}\left\{ t^\alpha D_{\alpha,\beta}(x,y,z)\right\} = (xy)^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right].$$

Applying the inversion formula of generalized Hankel-Clifford transformation to (4)

$$\int_0^\infty t^\alpha \left\{ t^{-\alpha-\beta} (zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right]\right\} D_{\alpha,\beta}(x,y,z)\,dz$$
$$= (xy)^{-\alpha-\beta} (xt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{xt}\right] (yt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{yt}\right]$$

and hence the result. In particular, taking $t = 0$, gives

ii) $$\int_0^\infty t^\alpha\, t^{-\alpha-\beta} (zt)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zt}\right] D_{\alpha,\beta}(x,y,z)\,dz = 1, \quad (9)$$

i.e. for which $x, y > 0, D_{\alpha,\beta}(x,y,z)$ belongs to $L^1_{0,\alpha,\beta}(0,\infty)$.

iii) $0 < x, y, z < \infty, D_{\alpha,\beta}(x,y,z) \ge 0$.

iv) $D_{\alpha,\beta}(x,y,z) = D_{\alpha,\beta}(y,x,z) = D_{\alpha,\beta}(z,x,y) = \ldots$

The generalized Hankel-Clifford translation T_y of $\phi \in L_p(0,\infty), 1 \le p \le \infty$, is defined by

$$T_y\phi(x) = \phi(x,y) = \int_0^\infty \phi(z) D_{\alpha,\beta}(x,y,z)\,dz, \quad 0 < x, y < \infty. \quad (10)$$

In [7], shown that the integral is convergent for almost all y and for each fixed x. For fixed x,

$$\left\|\phi(x,.)\right\|_p \le \left\|\phi\right\|_p. \quad (11)$$

The map $y \to T_y\phi$ is continuous from $(0,\infty)$ into $(0,\infty)$.

Let $p,q,r\in[1,\infty)$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1.$ The generalized Hankel-Clifford convolution of $\phi\in L_p(0,\infty)$ and $\psi\in L_p(0,\infty)$ is defined by

$$(\phi\#\psi)(x)=\int_0^\infty \phi(x,y)\psi(y)\,dy. \tag{12}$$

In [4] the integral is convergent for almost all $x, 0<x<\infty$ and

$$\|\phi\#\psi\|_r\le\|\phi\|_p\,\|\psi\|_q\,. \tag{13}$$

Moreover, $p=\infty,$ then $(\phi\#\psi)(x)$ is defined for all $x, 0<x<\infty$ and is continuous.

If $\phi,\psi\in L_1(0,\infty)$, then

$$(\phi\#\psi)\wedge(t)=\hat{\phi}(t)\hat{\psi}(t),\ 0\le t<\infty. \tag{14}$$

In this paper, in terms of the aforesaid generalized Hankel-Clifford translation T_y and dilation D_a defined by

$$D_{\alpha,\beta,a}\phi(x,y)=a^{2\beta-3/2}\phi(x/a,y/a), \tag{15}$$

is a continuous generalized Hankel-Clifford wavelet transformation is defined. Its continuity and boundedness properties are established. An inversion formula is obtained.

2. Continuous generalized Hankel-Clifford wavelet transformation

Let $\psi\in L_p(0,\infty), 1\le p<\infty$ be given. For $b\ge 0$ and $a>0$ define the generalized Hankel-Clifford wavelet transformation

$$\psi_{b,a}(x)=D_{\alpha,\beta,a}T_y\psi(x)=D_{\alpha,\beta,a}\psi(b,x)=a^{2\beta-3/2}\psi(b/a,x/a) \tag{16}$$

$$=a^{2\beta-3/2}\int_0^\infty D_{\alpha,\beta}(b/a,x/a,z)\psi(z)\,dz, \tag{17}$$

the integral being convergent by virtue of (11).

Using the wavelet $\psi_{b,a}$, we now define the generalized Hankel-Clifford wavelet transformation.

$$H_{1,\alpha,\beta}(b,a)=(H_{1,\alpha,\beta,\psi}f)(b,a)$$

$$=\langle f(t),\psi_{b,a}(t)\rangle$$

$$=\int_0^\infty f(t)\overline{\psi_{b,a}(t)}\,dt \tag{18}$$

$$=a^{2\beta-3/2}\int_0^\infty\int_0^\infty f(t)\overline{\psi(z)}D_{\alpha,\beta}(b/a,t/a,z)\,dz\,dt \tag{19}$$

provided the integral is convergent. The continuity of the generalized Hankel-Clifford wavelet follows from the boundedness property of the generalized Hankel-Clifford translation [5].

Lemma 1. Let $\psi \in L_p(0,\infty), 1 \le p < \infty.$ Then for $y \ge 0,$ the map $y \to T_y f$ is continuous from $L_p(0,\infty)$ into $L_p(0,\infty)$. The function $\psi_{b,a}$ is defined almost everywhere on $[0,\infty)$, and $\|\psi_{b,a}(x)\|_p \le a^{(2\beta-3/2)(1/p-1)}\|\psi\|_p$. The existence of the generalized Hankel-Clifford transformation is given by the following theorem.

Theorem 2. Let $f \in L_p(0,\infty)$ and $\psi \in L_p(0,\infty)$ with $1 \le p, q < \infty$ and $\frac{1}{p}+\frac{1}{q}=1; H_{1,\alpha,\beta}(b,a)=(H_{1,\alpha,\beta,\psi} f)(b,a)$ be the continuous wavelet transform (18). Then

1) $(H_{1,\alpha,\beta} f)(b,a)$ is continuous on $(0,\infty)\times(0,\infty)$,
2) $\left\|\left(\left(H_{1,\alpha,\beta,\psi} f\right)(b,a)\, f\right)(b,a)\right\|_r \le a^{-2\beta+3/2}\|f\|_p\|\psi\|_q, \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1, 1 \le p,q,r<\infty,$
3) $\left\|\left(\left(H_{1,\alpha,\beta,\psi} f\right)(b,a)\, f\right)(b,a)\right\|_\infty \le a^{(-2\beta+3/2)\left(\frac{1}{q}-1\right)}\|f\|_p\|\psi\|_q, \frac{1}{p}+\frac{1}{q}=1.$

Proof.

1) Let (b_0,a_0) be an arbitrary but fixed point in $(0,\infty)\times(0,\infty)$. Then by Hölder's inequality,

$$\left|(H_{1,\alpha,\beta} f)(b,a)-(H_{1,\alpha,\beta} f)(b_0,a_0)\right|$$

$$\le a^{2\beta-3/2}\int_0^\infty\int_0^\infty \left| f(t)\psi(z)\left[D_{\alpha,\beta}(b/a,t/a,z)-D_{\alpha,\beta}(b_0/a_0,t/a_0,z)\right]\right| dtdz$$

$$\le a^{2\beta-3/2}\left[\int_0^\infty\int_0^\infty |f(t)|^p \left|D_{\alpha,\beta}(b/a,t/a,z)-D_{\alpha,\beta}(b_0/a_0,t/a_0,z)\right| dtdz\right]^{1/p}$$

$$\times\left[\int_0^\infty\int_0^\infty |\psi(z)|^q \left|D_{\alpha,\beta}(b/a,t/a,z)-D_{\alpha,\beta}(b_0/a_0,t/a_0,z)\right| dtdz\right]^{1/q}$$

Since by (9), $\int_0^\infty \left|D_{\alpha,\beta}(b/a,t/a,z)-D_{\alpha,\beta}(b_0/a_0,t/a_0,z)\right| dt \le 2$, by dominated convergence theorem and continuity of $D_{\alpha,\beta}(b/a,t/a,z)$ in the variables b and a, we have $\lim_{\substack{b\to b_0\\ a\to a_0}}\left|(H_{1,\alpha,\beta} f)(b,a)-(H_{1,\alpha,\beta} f)(b_0,a_0)\right|=0$. This proves that $H(b,a)$ is continuous on $(0,\infty)\times(0,\infty)$.

2) Inequality (13) proves $\left\|\left(\left(H_{1,\alpha,\beta,\psi} f\right)(b,a)\, f\right)(b,a)\right\|_r \le a^{-2\beta+3/2}\|f\|_p\|\psi\|_q, \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1, 1 \le p,q,r<\infty.$

3) It can be proved using Hölder's inequality.

3. An inversion formula

In this section, it is shown that the function f can be recovered from its wavelet transform when the wavelet ψ satisfies admissibility condition.

Theorem 3. Let $\psi \in L^2(\mathrm{R}_+)$ be a basic wavelet which defines generalized Hankel-Clifford wavelet transformation (18). Then, for

$$A_\psi = \int_0^\infty w^{2\beta-3/2} \left|\hat{\psi}(w)\right|^2 dw > 0, \tag{20}$$

we have

$$\int_0^\infty\int_0^\infty \Big(\big(H_{1,\alpha,\beta,\psi} f\big)(b,a)\, f\Big)(b,a)\overline{\Big(\big(H_{1,\alpha,\beta,\psi} f\big)(b,a)\, g\Big)(b,a)}\; a^{2\beta-3/2} da db = A_\psi \langle f, g\rangle \tag{21}$$

for all $f, g \in L^2(\mathrm{R}_+)$

Proof. In view of representation (7),

$$\big(H_{1,\alpha,\beta,\psi} f\big)(b,a)$$

$$= \int_0^\infty\int_0^\infty f(t)\overline{\psi(z)} D_{\alpha,\beta}(b/a, t/a, z)\, dz dt$$

$$= a^{2\beta-3/2}\int_0^\infty\int_0^\infty \hat{f}(x/a)\overline{\psi(z)}\left[\begin{matrix}\left\{\left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{xb}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{xb}{a}}\right]\right\} \\ \left\{z^{-\alpha-\beta}(zax)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zax}\right]\right\}\end{matrix}\right] dx dz$$

$$= a^{2\beta-3/2}\int_0^\infty \hat{f}(x/a)\overline{\hat{\psi}(x)}\left\{\left\{\left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{xb}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{xb}{a}}\right]\right\}\right\} dx$$

$$= \int_0^\infty \hat{f}(u)\overline{\hat{\psi}(au)}\left\{b^{-\alpha-\beta}(bu)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bu}\right]\right\} du$$

$$= \left(\hat{f}(u)\overline{\hat{\psi}(au)}\right)\wedge(b). \tag{22}$$

Parseval identity (6) yields

$$\int_0^\infty \big(H_{1,\alpha,\beta,\psi} f\big)(b,a)\overline{\big(H_{1,\alpha,\beta,\psi} f\big)(b,a)}\, db$$

$$= \int_0^\infty \left(\hat{f}(u)\overline{\hat{\psi}(au)}\right)\wedge(b)\overline{\left(\hat{g}(u)\overline{\hat{\psi}(au)}\right)}\wedge(b)\; db$$

$$= \int_0^\infty \left(\hat{f}(u)\overline{\hat{\psi}(au)}\right)\overline{\left(\hat{g}(u)\overline{\hat{\psi}(au)}\right)}\; du.$$

Now multiplying by $a^{2\beta-3/2} da$ and integrating, the results obtained are

$$\int_0^\infty\int_0^\infty \big(H_{1,\alpha,\beta,\psi} f\big)(b,a)\overline{\big(H_{1,\alpha,\beta,\psi} f\big)(b,a)}\, db\;\; a^{2\beta-3/2} da$$

$$= \int_0^\infty\int_0^\infty \left(\hat{f}(u)\overline{\hat{\psi}(au)}\right)\overline{\left(\hat{g}(u)\overline{\hat{\psi}(au)}\right)}\; du.\; a^{2\beta-3/2} da$$

$$= \int_0^\infty \hat{f}(u)\overline{\hat{g}(u)}du \int_0^\infty \hat{\psi}(au)\overline{\hat{\psi}(au)} \,.\, a^{2\beta-3/2}da$$

$$= \int_0^\infty \hat{f}(u)\overline{\hat{g}(u)}du \int_0^\infty \left|\hat{\psi}(au)\right|^2 \,.\, a^{2\beta-3/2}da$$

$$= \int_0^\infty \hat{f}(u)\overline{\hat{g}(u)}du \int_0^\infty \left|\hat{\psi}(w)\right|^2 \,.\, w^{2\beta-3/2}dw$$

$$= A_\psi \langle f, g\rangle.$$

Notice that admissibility condition (20) requires that $\hat{\psi}(0)=0$. If $\hat{\psi}$ is continuous then from (3) it follows that

$\int_0^\infty \psi(x)dx = 0.$ This justifies the wavelet to the function.

Now consider

$\psi\left(\frac{b}{a},\frac{t}{a}\right)=\psi(x,y)$ by putting $\frac{b}{a}=x$ and $\frac{t}{a}=y$. Then $\psi(x,y)=\int_0^\infty \psi(z)D_{\alpha,\beta}(b/a,t/a,z)dz.$

Since

$$D_{\alpha,\beta}(x,y,z)=\int_0^\infty \xi^{-\alpha}\begin{bmatrix} x^{-\alpha-\beta}(x\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x\xi}\right] \\ y^{-\alpha-\beta}(y\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{y\xi}\right] z^{-\alpha-\beta}(z\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{z\xi}\right]d\xi \end{bmatrix}$$

By Substituting

$$\psi(x,y)=\int_0^\infty \psi(z)\left[\int_0^\infty \xi^{-\alpha}\begin{Bmatrix} x^{-\alpha-\beta}(x\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x\xi}\right] \\ y^{-\alpha-\beta}(y\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{y\xi}\right] \\ z^{-\alpha-\beta}(z\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{z\xi}\right]d\xi \end{Bmatrix}\right]dz$$

$$=\int_0^\infty\left(\int_0^\infty \psi(z)z^{-\alpha-\beta}(z\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{z\xi}\right]dz\right)\xi^{-\alpha}\begin{Bmatrix} x^{-\alpha-\beta}(x\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x\xi}\right] \\ y^{-\alpha-\beta}(y\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{y\xi}\right] \end{Bmatrix}d\xi$$

$$=\int_0^\infty \xi^{-\alpha}\left\{x^{-\alpha-\beta}(x\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x\xi}\right]y^{-\alpha-\beta}(y\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{y\xi}\right]\right\}\left(F_{1,\alpha,\beta}\psi\right)(\xi)d\xi$$

Substitute $\frac{b}{a}=x$ and $\frac{t}{a}=y.$

$$\psi\left(\frac{b}{a},\frac{t}{a}\right)=\int_0^\infty \xi^{-\alpha}\left\{\left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{b\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{b\xi}{a}}\right]\left(\frac{t}{a}\right)^{-\alpha-\beta}\left(\frac{t\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{t\xi}{a}}\right]\right\}\left(F_{1,\alpha,\beta}\psi\right)(\xi)d\xi.$$

$$\left(H_{1,\alpha,\beta,\psi} f\right)(b,a)$$
$$= a^{2\beta-3/2}\int_0^\infty \psi\left(\frac{b}{a},\frac{t}{a}\right) f(t)\,dt$$
$$= a^{2\beta-3/2}\int_0^\infty \left(\int_0^\infty \xi^{-\alpha}\left\{\begin{matrix}\left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{b\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{b\xi}{a}}\right] \\ \left(\frac{t}{a}\right)^{-\alpha-\beta}\left(\frac{t\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{t\xi}{a}}\right]\end{matrix}\right\}\left(F_{1,\alpha,\beta}\psi\right)\right)\times f(t)\,dt.$$
$$= a^{2\beta-3/2}\int_0^\infty \left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{b\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{b\xi}{a}}\right]\left(\int_0^\infty \left(\frac{t}{a}\right)^{-\alpha-\beta}\left(\frac{t\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{t\xi}{a}}\right] f(t)\,dt\right)\times \xi^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(\xi)\,d\xi$$
$$= a^{2\beta-3/2}\int_0^\infty \left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{b\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{b\xi}{a}}\right]\left(F_{1,\alpha,\beta} f\right)\left(\frac{\xi}{a}\right)\times \xi^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(\xi)\,d\xi.$$

By substituting $\frac{\xi}{a} = x; d\xi = a dx$, the continuous generalized Hankel-Clifford wavelet transform can be written as

$$\left(H_{1,\alpha,\beta,\psi} f\right)(b,a) = a^{2\beta-3/2}\int_0^\infty \left(\frac{b}{a}\right)^{-\alpha-\beta}\left(\frac{b\xi}{a}\right)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{\frac{b\xi}{a}}\right]\left(F_{1,\alpha,\beta} f\right)\left(\frac{\xi}{a}\right)\times \xi^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(\xi)\,d\xi.$$
$$= a^{2\beta-3/2}\int_0^\infty \left(\frac{b}{a}\right)^{-\alpha-\beta}(bx)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right]\left(F_{1,\alpha,\beta} f\right)(x)\times(ax)^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(ax)\,a\,dx$$
$$= a^{2\beta-3/2}a^{-\alpha+\alpha+\beta+1}\int_0^\infty b^{-\alpha-\beta}(bx)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right]\left(F_{1,\alpha,\beta} f\right)(x)\times x^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(ax)\,dx$$
$$= a^{3\beta-1/2}\int_0^\infty b^{-\alpha-\beta}(bx)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right]\left(F_{1,\alpha,\beta} f\right)(x)\; x^{-\alpha}\left(F_{1,\alpha,\beta}\psi\right)(ax)\,dx.$$

These examples are shown in [10], considering Continuous Hankel-Clifford wavelet transformation. In this paper, the author considers the same examples to study in detail about generalized Hankel-Clifford wavelet transformation to compare and interpret the results.

Example 1. Assume that $f(t) = t^{-\tau}; 0 < \tau < -2\beta + 3/2.$

Then the generalized Hankel-Clifford wavelet transforms of $t^{-\tau}$ is given by
$$\left(H_{1,\alpha,\beta,\psi}\left(t^{-\tau}\right)\right)(b,a) = a^{2\beta-3/2}\int_0^\infty t^{-\tau}\psi\left(\frac{b}{a},\frac{t}{a}\right)dt$$

$$= a^{2\beta-3/2}\int_0^\infty \psi(z)\left(\int_0^\infty t^{-\tau} D_{\alpha,\beta}(b/a, t/a, z)\,dt\right)dz. \tag{23}$$

Since

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty \xi^{-\alpha}\left[\begin{matrix}x^{-\alpha-\beta}(x\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{x\xi}\right] \\ y^{-\alpha-\beta}(y\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{y\xi}\right] z^{-\alpha-\beta}(z\xi)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{z\xi}\right]d\xi\end{matrix}\right]$$

Using the representation (7) the inner integral can be evaluated by means of the formula

$$\int_0^\infty t^{-\tau} D_{\alpha,\beta}\left(b/a,t/a,z\right)dt$$

$$=\int_0^\infty t^{-\tau}\int_0^\infty (ax)^{-\alpha}\begin{bmatrix}\left(\frac{b}{a}\right)^{-\alpha-\beta}(bx)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right]\\ \left(\frac{t}{a}\right)^{-\alpha-\beta}(tx)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{tx}\right]\\ z^{-\alpha-\beta}(zax)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{zax}\right]\end{bmatrix} adxdt$$

$$=\int_0^\infty a^{-3\alpha-2\beta+1}x^{(\alpha+3\beta)/2}(zab)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right] J_{\alpha-\beta}\left[2\sqrt{zax}\right]\left(\int_0^\infty t^{-\tau-(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{tx}\right]dt\right)dx$$

$$=\int_0^\infty a^{-3\alpha-2\beta+1}x^{(\alpha+3\beta)/2}(zab)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left[2\sqrt{bx}\right] J_{\alpha-\beta}\left[2\sqrt{zax}\right]\left(\frac{x^{\tau-1+(\alpha+\beta)/2}\Gamma(1-\beta-\tau)}{\Gamma(\tau+\alpha)}\right)dx$$

$$=\frac{\Gamma(1-\beta-\tau)}{\Gamma(\tau+\alpha)}(zab)^{(\alpha+\beta)/2} a^{-3\alpha-2\beta+1}\int_0^\infty x^{\tau-1+\alpha+2\beta} J_{\alpha-\beta}\left[2\sqrt{bx}\right] J_{\alpha-\beta}\left[2\sqrt{zax}\right]dx$$

Now evaluating the last integral by calculations with MATLAB, and substituting the value of the integral (23) as in [9], we get

$$\left(H_{1,\alpha,\beta,\psi}\left(t^{-\tau}\right)\right)(b,a)=\frac{\Gamma(2\alpha+\beta+\tau)\Gamma(1-\beta-\tau)}{\Gamma(\tau+\alpha)\Gamma(-\alpha-2\beta-\tau+1)}a^{(-7\alpha+\beta-1)/2}b^{-\beta/2}$$
$$\times\int_0^\infty z^{(-\alpha+\beta)/2}(za-b)^{-\alpha-2\beta-\tau}\ LegendreP\left(\tau-1+\alpha+2\beta,-\alpha+\beta,\frac{b+za}{za-b}\right)\psi(z)\,dz.$$

for $0<\tau<-2\beta+3/2, a,b>0.$

Example 2. In this example we consider the generalized Hankel-Clifford wavelet transformation of order zero. Assume that $f(t)=t^{-3}\sin(w_0t); w_0>0.$

Then we have

$$\left(H_{1,\alpha,\beta,\psi}\left(t^{-3}\sin(w_0t)\right)\right)(b,a)$$

$$=a^{2\beta-3/2}\int_0^\infty\left(t^{-3}\sin(w_0t)\right)\psi\left(\frac{b}{a},\frac{t}{a}\right)dt$$

$$=a^{2\beta-3/2}\int_0^\infty\psi(z)\left(\int_0^\infty\left(t^{-3}\sin(w_0t)\right)D_{\alpha,\beta}\left(b/a,t/a,z\right)dt\right)dz. \qquad (24)$$

Using the representation (7) the inner integral can be evaluated by means of the formula

$$\int_0^\infty t^{-3}\sin(w_0 t) D_{\alpha,\beta}(b/a, t/a, z)\,dt$$

$$=\int_0^\infty t^{-3}\sin(w_0 t)\int_0^\infty (ax)^{-\alpha}\begin{bmatrix}\left(\frac{b}{a}\right)^{-\alpha-\beta}(bx)^{(\alpha+\beta)/2} J_0\left[2\sqrt{bx}\right]\left(\frac{t}{a}\right)^{-\alpha-\beta}(tx)^{(\alpha+\beta)/2} J_0\left[2\sqrt{tx}\right] \\ z^{-\alpha-\beta}(zax)^{(\alpha+\beta)/2} J_0\left[2\sqrt{zax}\right]\end{bmatrix} adxdt$$

$$=\int_0^\infty a^{-3\alpha-2\beta+1}x^{(\alpha+3\beta)/2}(zab)^{(\alpha+\beta)/2} J_0\left[2\sqrt{bx}\right] J_0\left[2\sqrt{zax}\right]$$

$$\times\left(\int_0^\infty t^{-3-(\alpha+\beta)/2}\sin(w_0 t) J_0\left[2\sqrt{tx}\right]dt\right)dx$$

$$=\int_0^\infty a^{-3\alpha-2\beta+1}x^{(\alpha+3\beta)/2}(zab)^{(\alpha+\beta)/2} J_0\left[2\sqrt{bx}\right] J_0\left[2\sqrt{zax}\right]$$

$$\times 2^{-3-\alpha-\beta/2}\sqrt{\pi}\; w_0^{2+\alpha+\beta/2}\begin{bmatrix}\dfrac{\Gamma\left(-\dfrac{2+2\alpha+\beta}{4}\right)}{\Gamma\left(2+\dfrac{2\alpha+\beta}{4}\right)}F\left[-1-\dfrac{\alpha}{2}-\dfrac{\beta}{4},-1-\dfrac{\alpha}{2}-\dfrac{\beta}{4},\left[\dfrac{1}{2},\dfrac{1}{2},1\right],-\dfrac{x^2}{4w_0^2}\right] \\ -2\dfrac{\Gamma\left(\dfrac{-2\alpha-\beta}{4}\right)}{w_0\Gamma\left(\dfrac{6+2\alpha+\beta}{4}\right)}x\,F\left[-\dfrac{\alpha}{2}-\dfrac{\beta}{4},-\dfrac{1}{2}-\dfrac{\alpha}{2}-\dfrac{\beta}{4},\left[1,\dfrac{3}{2},\dfrac{3}{2}\right],-\dfrac{x^2}{4w_0^2}\right]\end{bmatrix}dx$$

Now evaluating the last integral by calculations using MATLAB, and substituting the value of the integral (24) by, we get

$$\left(H_{1,\alpha,\beta,\psi}\left(t^{-3}\sin(w_0 t)\right)\right)(b,a) = a^{-5\alpha/2-1/2}b^{\beta/2}w_0\,\frac{\sin\left(\pi(3+\alpha+2\beta)\right)\left[\Gamma(3+\alpha+2\beta)\right]^2}{\sin\left(\pi(3+(\alpha+\beta)/2)\right)\left[\Gamma(3+(\alpha+\beta)/2)\right]^2}$$

$$\int_0^\infty z^{(\alpha+\beta)/2}\,(za-b)^{-\alpha-2\beta-3}\,LegendreP\left(\alpha+2\beta+2,\frac{za+b}{za-b}\right)\psi(z)\,dz$$

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